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Grain size distribution and flow stress in tectonites

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Abstract—Flow stresses in dynamically recrystallized tectonites are usually determined by using empirically calibrated grain size–stress relations. As grain size adjusts locally to stress, the validity of the procedure is dependent on the assumption that the local stress, at grain or subgrain level, is equal to the externally applied tectonic stress. The local stress, however, is a stochastic variable with a distribution related to the tectonic stress: once this fact is recognised, the question becomes that of deciding which measure of grain size, and therefore of local stress, gives the best estimate of the tectonic stress.

Current procedures implicitly assume that such a measure is the mean grain size. It is shown here that, on the basis of the most general probabilistic considerations, the local stress, and therefore the grain size, can be expected to have a lognormal distribution, and consequently that the median grain size, and not the mean, is the best indicator of tectonic stress. The lognormality of grain size has been confirmed by observations, both on metals and on rocks.

The use of the mean, rather than the median, grain size introduces a further source of uncertainty in flow stress determinations. An expression for the error in stress is derived, and found to depend on the coefficients of variation (i.e. dispersions) in the grain size distribution of calibrating curve and field tectonite. If these two are the same (or in the trivial case in which they are both very small), no error arises from the use of mean grain size. But, if this condition is not fulfilled, an error of up to 10–20% in flow stress may occur.

INTRODUCTION

SYNTECTONICALLY recrystallized grain size is widely used to infer flow stresses in tectonites. Post (1977) and Mercier *et al.* (1977) showed that grain size varies with stress in experimentally deformed dunite; Twiss (1977) provided a model to account for the observations, based on dislocation energy density considerations. The underlying assumption is that, during steady-state dislocation creep, the back stress on a dislocation, caused by interaction with neighbouring dislocations, is equal to the externally applied (tectonic) stress. The relation between stress σ and recrystallized grain size d is

$$\sigma = \mathrm{B}d^{-p}, \qquad (1)$$

where $0.67 \leq p \leq 0.78$. Equation (1) has veen verified in a number of experimental studies (see e.g. Ross *et al.* 1980). Based on the laboratory calibration of the parameters B and d, it has been applied to shear zones, where the estimated stress is usually between 20 and 200 MPa (S. White 1979a, Etheridge & Wilkie 1979, 1981, Christie & Ord 1980, Kohlstedt & Weathers 1980, J. C. White 1982), and to upper mantle peridotites, where stresses cluster around the 10–20 MPa range (Nicolas 1978, Mercier 1980).

Several difficulties are associated with palaeostress estimates, and are reflected in the dispersion of results. Possible sources of uncertainty have been discussed by S. White (1979b), and include post-tectonic annealing, non-dislocation creep mechanisms, influence of a second mineral phase, temperature-dependence of grain size, and effect of water content. Furthermore, equation (1) holds for grain boundary migration (GBM) recrystallization only; the grain-size dependence of stress is different for subgrain rotation (SGR) recrystallization, which is favoured by low σ and/or low T and has a grain size exponent of unity (Poirier & Guillopé 1979, Mercier 1980, J. C. White 1982).

In this paper, another source of uncertainty is considered, namely, the possible difference between applied (tectonic) stress and local (grain and subgrain scale) stress. It will be shown that neglect of the statistical fluctuations of local stress (that is, neglect of the properties of its probability distribution) may lead to significant errors in palaeostress determinations. It is assumed that the local grain size adjusts to the local stress, and that therefore there is correspondence between the stress distribution and the grain size distribution. This local equilibration is required in Twiss's (1977) theory and confirmed by observation.

GRAIN SIZE DISTRIBUTION

Although detailed information is very seldom given on how 'average grain size' is arrived at, optical methods are usually some variant of the linear intercept method, as proposed by Smith & Guttman (1953). The average linear intercept, or average length (in thin section) between intersections of grain boundaries with a grid line, is taken as a measure of grain size. Etheridge & Wilkie (1979) estimated average linear intercepts in three mutually perpendicular directions, then obtained d by calculating the diameter of a sphere of equal volume. A similar procedure is followed in most cases, with some minor modifications. Sometimes information on measurement dispersion is also given, in terms of an error bar; but the form of the distribution of d is usually not investigated.

The grain size distribution, even in a monomineralic rock subject to uniform applied stress, has statistical properties that are neglected by the current procedures.

In metals, the grain size distribution is lognormal (Exner 1972), and so it is, approximately, in the metamorphic rocks examined by Kretz (1966). In the rare instances in which the complete grain distribution of a syntectonically recrystallized tectonite has been determined, it has been found to be approximately lognormal (Etheridge & Wilkie 1979, p. 466). All the available evidence suggests that the grain size distribution is not normal, and very likely to be lognormal.

There are also theoretical grounds to expect the grain size to be a lognormal variate. Neoblast size clearly reflects the local stress (see Etheridge & Wilkie 1981, fig. 11, p. 504 for a convincing example). The local stress is the outcome of a large number of mutually independent factors operating simultaneously, i.e. it is expressible as

$$\sigma_l = \sigma \prod_i \zeta_i \tag{2}$$

where σ is the applied (tectonic) stress and $\{\zeta_i, i = 1, 2, \ldots k\}$ are a set of independent stochastic variations. It follows by the central limit theorem that σ_l is lognormal. (See Aitchison & Brown 1957, pp. 20–27, for a more detailed discussion. Indeed, lognormality is the rule, rather than the exception, for stochastic variates which are non-negative and can be regarded as the outcome of a set of random variations: some of these variates, e.g. geochemical concentrations, particle size, fault length, are of geological interest; cf. Agterberg 1974, p. 206, Ranalli 1976, 1980).

The expected value of a lognormal variate of the form given by equation (2) is

$$E\{\ln \sigma_l\} = \ln \sigma + E\{\ln \prod_i \zeta_i\}$$

$$= \ln \sigma + E\{\sum_i \ln \zeta_i\}$$

and therefore, assuming $E\{\sum_i \ln \zeta_i\} = 0$ (since $\{\zeta_i, i = 1, 2, \ldots k,\}$ are a sequence of random departures from unity),

$$E\{\ln \sigma_l\} = \ln \sigma.$$

Consequently, the best estimate of applied stress is (a superscripted horizontal bar denotes estimation from the sample)

$$\bar{\sigma} = \exp(\bar{E}\{\ln \sigma_l\}) \tag{3}$$

where the term on the right-hand side is the median of the local stress σ_l (Aitchison & Brown 1957, p. 9). Since grain size equilibration is local, it follows that d is also a lognormal variate (as could be seen immediately from equation (1)), and its estimated median is

$$\bar{d}_m = \exp(\bar{E}\{\ln d\}) = \exp\left(\frac{1}{n}\sum_i \ln d_i\right)$$

$$= \exp\left(\frac{1}{n}\ln \prod_i d_i\right) = \left(\prod_i d_i\right)^{1/n}, \tag{4}$$

where n is the sample size.

Equation (4) shows that the geometric mean of d, and not the arithmetic mean, is the best measure of applied stress. The median is the 50th percentile of the distribution, and is not affected by the dispersion of $\ln d$. Only for symmetric distributions do median and mean coincide; for positive skewness, the arithmetic mean is an overestimation of the median.

It follows from the above line of reasoning that the relation between applied stress and neoblast size should read

$$\sigma = B' \, d_m^{-p'}, \tag{5}$$

where B', p' are the (unknown) relevant parameters. The coefficients B and p in equation (1) will not necessarily be the same as their primed equivalents in equation (5). An obvious solution to the problem would involve the experimental recalibration of the relation between stress and grain size in terms of d_m, and the use of d_m, B' and p' in the determination of palaeostresses in tectonites. Alternatively, one may try to estimate the error involved in the use of the arithmetic mean of grain size. The remainder of this paper deals with the latter problem.

ERROR ARISING FROM THE USE OF MEAN GRAIN SIZE

How the error arises is qualitatively depicted in Fig. 1. The curve denoted by e represents the laboratory-calibrated (d, σ)-in relation, obtained by applying a stress σ_e, and observing the corresponding mean grain size $\langle d_e \rangle$, which permits the determination of the parameters B and p in equation (1). (The arrows on the dashed lines symbolize the flow of the procedure.) But, since the applied stress, through the lognormally distributed local stress, is related to the median grain size d_m, and $d_m \leq \langle d_e \rangle$, relation (5) is represented by the line denoted by m. The vertical distance between e and m is a function of the difference between $\langle d_e \rangle$ and d_m, that is, of the dispersion of the distribution. The two curves are drawn parallel because it is assumed that $p = p'$, that is, on the average, the dispersion is not a function of stress. While this hypothesis is not proven, it remains the simplest, and departures from it are likely to represent only second-order effects.

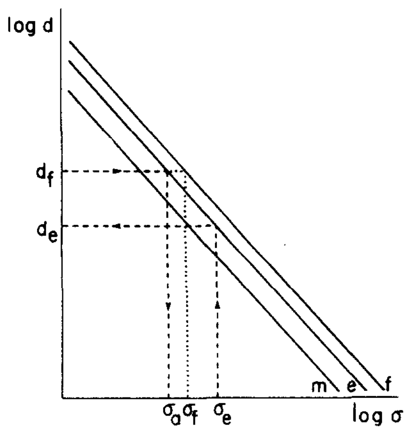

Fig. 1. Schematic diagram illustrating how a difference between σ_a and σ_f arises. See text for explanation.

Now consider the situation in which the mean grain size $\langle d_f \rangle$ is measured in a field sample. Using the calibration curve e, a palaeostress σ_a is inferred. This is correct only if the difference between mean and median is the same in the laboratory and in the field. If the field tectonite, for instance, has a distribution with larger dispersion, its relationship between stress and mean grain size will be represented by the curve f, and therefore the correct palaeostress is σ_f; use of the laboratory-calibrated coefficients leads to an underestimation of the stress. The opposite holds if the dispersion of the field sample is less than the average dispersion of the samples that have been used for the laboratory curve. The estimation is correct ($\sigma_a = \sigma_f$) only if the dispersion of field and laboratory samples is the same (which includes negligible skewness as a particular case).

The above is simply a statement of the fact that the relation between mean and median grain size is not necessarily unique in different rocks, or, in other words, rocks may have the same mean recrystallized grain size and have been subjected to different flow stresses, or vice-versa. The mean grain size, and hence the coefficient B in equation (1), is affected by the dispersion of the distribution, and its accurate use would require the knowledge of this dispersion.

The previous argument can be formalized, and an estimate of the stress error given as a function of the dispersion of the grain size distribution, as follows: equating equations (1) and (5) under a given stress [the symbols B_e, $\langle d_e \rangle$ are used to emphasize that it is the mean grain size, obtained from experimental calibration, which appears in (1)]

$$\sigma = B_e \langle d_e \rangle^{-p} = B' d_m^{-p'}$$

and under the hypothesis $p = p'$, one obtains

$$\frac{\langle d_e \rangle}{d_m} = \left(\frac{B_e}{B'}\right)^{1/p}.$$

Similarly, for the field tectonite (note that B_f is unknown and not necessarily equal to B_e),

$$\frac{\langle d_f \rangle}{d_m} = \left(\frac{B_f}{B'}\right)^{1/p}.$$

Recalling the relation between the mean of the untransformed linear variate and the mean E and variance D^2 of the lognormal distribution (Aitchison & Brown 1957, p. 8),

$$\langle d \rangle = \exp\,(E + D^2/2) = d_m\,(\eta^2 + 1)^{1/2},$$

where η is the coefficient of variation (ratio of standard deviation to mean), and use has been made of the relations

$$d_m = \exp\,(E)$$
$$\eta = [\exp\,(D^2) - 1]^{1/2},$$

it follows that

$$B_e = B'\,(\eta_e^{\,2} + 1)^{p/2}$$
$$B_f = B'\,(\eta_f^{\,2} + 1)^{p/2}, \qquad (6)$$

where η_e, η_f are the coefficients of variation in the laboratory (actually, the average of several experimentally deformed samples) and in the field, respectively.

Equations (6) express explicitly the already noted dependence of the parameter B in equation (1) on the dispersion of the grain size distribution, and yield immediately the ratio $\langle d_e \rangle / \langle d_f \rangle$ for a given stress as a function of η_e and η_f.

The error in the flow stress as determined from the mean grain size in the field stems from using the laboratory calibration for B, i.e. B_e rather than B_f. Terming σ_a the 'apparent' stress thus determined, and σ_f the 'correct' (unknown) stress,

$$\sigma_a = B_e \langle d_f \rangle^{-p} = B'(\eta_e^{\,2} + 1)^{p/2} \langle d_f \rangle^{-p}$$
$$\sigma_f = B_f \langle d_f \rangle^{-p} = B'(\eta_f^{\,2} + 1)^{p/2} \langle d_f \rangle^{-p}$$

their ratio is

$$\frac{\sigma_f}{\sigma_a} = \left(\frac{\eta_f^2 + 1}{\eta_e^2 + 1}\right)^{p/2}. \qquad (7)$$

Values of the ratio σ_f/σ_a as calculated from equation (7) for different values of η_e and $\gamma = \eta_f/\eta_e$ are shown in Table 1. Obviously, for $\gamma = 1$, $\sigma_f = \sigma_a$, no matter what is the value of η_e. The ranges shown probably cover most of the cases of interest. A distribution with $\eta = 0.5$ has noticeable skewness; for $\eta < 0.25$, the skewness is undetectable. (Kretz 1966 found $\eta \simeq 0.5$).

Figure 2 is a graphical representation of the results shown in Table 1. Only values in the lower range of η_e and in the central range of γ are probably realistic; errors in stress of up to 20% of the apparent value are quite

Table 1. Values of σ_f/σ_a for different values of the coefficients of variation.

$\gamma = \eta_f/\eta_e$ \\ η_e	0.2	0.3	0.4	0.5	0.6	0.8	1.0
0.50	0.990	0.978	0.963	0.945	0.925	0.886	0.848
0.75	0.994	0.987	0.978	0.968	0.958	0.937	0.917
1.00	1	1	1	1	1	1	1
1.25	1.008	1.016	1.026	1.038	1.050	1.072	1.091
1.50	1.017	1.035	1.057	1.081	1.124	1.149	1.185
1.75	1.027	1.057	1.092	1.128	1.165	1.230	1.281
2.00	1.040	1.081	1.129	1.179	1.227	1.312	1.378

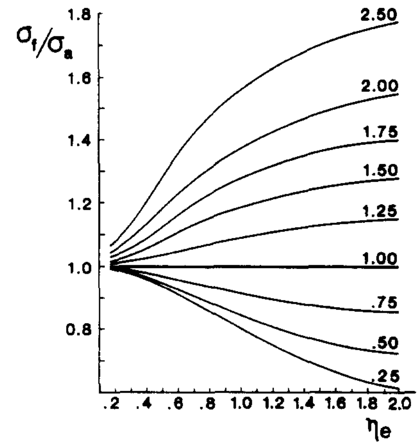

Fig. 2. Error in stress (σ_t/σ_a) as a function of η_e and γ; the latter's values are shown on the curves.

possible, if circumstances are unfavourable. On the other hand, errors may be negligible, if both η_e and η_f are very low (note that, since results are given for a fixed γ, a decrease in η_e implies a decrease in η_f), or if the ratio of the two coefficients of variation is always close to unity. The question of the actual magnitude of the error in individual cases cannot be answered without a knowledge of the coefficients of variation.

CONCLUSIONS

By considering the most likely statistical properties of local (grain-scale) stress distribution in tectonites, and assuming that neoblast size equilibrates to the local stress, it can be inferred that both are lognormal variates. This conclusion has been confirmed in the few instances in which the distribution of recrystallized grain size has been determined.

Since in grain size determination one deals with an untransformed linear variate, the best estimate of central tendency is not the mean, but the median (geometric mean), which is unaffected by the dispersion of the lognormal distribution. Furthermore, it can be proven that the median grain size, and not the mean grain size, is directly related to the applied (tectonic) stress. In principle, therefore, both the calibrating curve and the measurements on field tectonites should relate applied stress to median neoblast size. This would require no additional labour, since the linear intercept method can be equally applied to the determination of the median.

If the (arithmetic) mean grain size is used, an error in the estimated flow stress may be introduced. This error is a function of the coefficients of variation of the grain size distribution in the calibrating samples and in the field tectonites. When the two coefficients of variation are the same, no error arises; also, the error is negligibly small if the coefficients of variation are less than about 1/4. When neither of these conditions is fulfilled, the actual tectonic stress could differ from the apparent (estimated) tectonic stress by 10–20%. Much larger errors are possible, but they call for unrealistically large coefficients of variation and ratios. On the other hand, errors of up to the above range could arise if, say, $\eta_e \simeq 0.4$–0.6 and $\eta_f/\eta_e \simeq 0.50$–$1.50$, which *a priori* are quite realistic values.

Only the possibility of errors in flow stress arising from the use of the mean neoblast size has been established: it may well be that, in particular instances, the errors are negligible. The main point is that neglect of the statistical properties of the grain size distribution, coupled with the use of the arithmetic mean, may be a risky procedure. It would be worthwhile to carry out a systematic study of the dispersion of grain size in dynamically recrystallized tectonites, to set the matter to rest, or to modify our methods as necessary.

The type of error discussed here is inherent in the assumptions and procedures used in tectonic stress determinations, and is additional to the other errors and uncertainties which have been discussed in the literature.

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